Tayebian Duality and Selector-Matrix Functoriality: A Unified Research Paper Extending Prime-Height Tetration, $X(x)^{X(x)}$ for Infinite Polynomials, and Discrete Selectors

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Abstract

We consolidate and extend a unified analytic program developed in prior sections: (i) prime-height tetration ${}^{p}x$ as a generator hierarchy extending x^x; (ii) exact tower calculus via derivative and Jacobian ladders; (iii) continuous-height extension via Abel linearization $T(x,h)$ and super-logarithm slog_x; (iv) mode coordinates, residue-comb selectors, and modular phase extraction. We then add a structurally higher layer: the Tayebian framework as a dual representation that contains both a Hilbert-like state description and a Lagrangian-like generator description. Within this dual structure we construct a functor that maps Tayebian objects to selector-matrix algebras acting on Hilbert projections. This functorial viewpoint formally places selector matrices as representation-dependent shadows (endomorphisms on a chosen basis) of a higher, representation-independent Tayebian object, and it yields commutative diagrams linking (a) action/generator evolution, (b) spectral/mode reconstructions, and (c) discrete selection operations. All derivations are exact and non-asymptotic.

Contents

1 Introduction and Scope

We consider nonlinear analytic generators of the form x^x and their generalizations:

- **Prime-height tetration:** ${}^{p}x$ for primes p, realized as right-associated power towers.
- **Continuous-height extension:** $T(x,h)$ satisfying $T(x,h+1) = x^{T(x,h)}$, interpolating integer heights.
- **Infinite-polynomial substitution:** $F(x) = X(x)^{X(x)}$ where $X(x) = \sum_{n\geq 0} a_n x^n$ is a formal/analytic infinite polynomial.
- **Selectors:** residue-comb operators and selector matrices used to extract/discretize components.

The present paper adds a structural result: *selector matrices arise functorially from Tayebian objects after Hilbert projection and basis choice.* This establishes a strict hierarchy:

$$\text{Tayebian object} \implies \text{Hilbert projection} \implies \text{basis} \implies \text{selector matrices.}$$

We formalize this chain as a functor from a Tayebian category to a matrix/endomorphism category.

2 Prime-Height Tetration and Exact Tower Calculus

2.1 Right-associated towers and prime heights

Define the height-n tower recursively:

$$t_1(x) = x, \qquad t_{n+1}(x) = x^{t_n(x)}. \tag{1}$$

Then ${}^{n}x = t_n(x)$. Prime-height tetration is the restriction:

$${}^{p}x := t_p(x) \qquad \text{for prime } p.$$

2.2 Log ladder and derivative ladder

For $x > 0$, write

$$t_{n+1}(x) = \exp\big(t_n(x)\log x\big).$$

Define the log ladder $\ell_n(x) = \log t_n(x)$, giving

$$\ell_{n+1}(x) = t_n(x)\log x. \tag{2}$$

Differentiate to obtain the exact derivative recursion

$$t'_{n+1}(x) = t_{n+1}(x)\Big(t'_n(x)\log x + \frac{t_n(x)}{x}\Big). \tag{3}$$

Define the log-derivative $D_n(x) = t'_n(x)/t_n(x)$. Using (2) yields

$$\boxed{D_{n+1}(x) = D_n(x)\,\ell_{n+1}(x) + \frac{t_n(x)}{x}, \qquad D_1(x) = \frac{1}{x}.} \tag{4}$$

Hence

$$\boxed{t'_p(x) = t_p(x)\,D_p(x)} \qquad \text{(in particular for prime } p\text{).} \tag{5}$$

2.3 Tower-induced pushforward measure

On a domain where t_p is invertible, let $\tau_p = t_p^{-1}$. With $u = t_p(x)$ we have

$$du = t'_p(x)\,dx = u\,D_p(\tau_p(u))\,dx,$$

thus

$$\boxed{dx = \frac{du}{u\,D_p(\tau_p(u))}.} \tag{6}$$

For integrable Φ and W,

$$\boxed{\int \Phi(t_p(x))\,W(x)\,dx = \int \Phi(u)\,W(\tau_p(u))\,\frac{du}{u\,D_p(\tau_p(u))}.} \tag{7}$$

3 Abel Linearization and Continuous Height

Fix $x > 0$ and define $F_x(y) = x^y$. An Abel function A_x satisfies

$$\boxed{A_x(F_x(y)) = A_x(y) + 1} \qquad \Leftrightarrow \qquad A_x(x^y) = A_x(y) + 1. \tag{8}$$

Assuming A_x invertible on a suitable domain, define the continuous-height tetration

$$\boxed{T(x,h) = A_x^{-1}(A_x(1) + h).} \tag{9}$$

Then

$$T(x,h+1) = x^{T(x,h)}, \qquad T(x,0) = 1, \qquad T(x,1) = x,$$

and for integers $n \geq 0$,

$$\boxed{T(x,n) = t_n(x) = {}^n x.} \tag{10}$$

Therefore prime-height tetration is sampling:

$$\boxed{{}^{p}x = T(x,p) \qquad (p \text{ prime}).} \tag{11}$$

Define the super-logarithm coordinate

$$\boxed{\operatorname{slog}_x(y) = A_x(y) - A_x(1),} \tag{12}$$

which satisfies $\operatorname{slog}_x(x^y) = \operatorname{slog}_x(y) + 1$ and $\operatorname{slog}_x(T(x,h)) = h$.

4 Infinite Polynomial Substitution: $X(x)^{X(x)}$

Let

$$X(x) = \sum_{n=0}^{\infty} a_n x^n$$

as an analytic function in a neighborhood or as a formal series in $\mathbb{C}[[x]]$.

4.1 Canonical definition via exponential-log

Choose a branch of (analytic) or use the formal logarithm (if $a_0 \neq 0$). Define

$$\boxed{F(x) := X(x)^{X(x)} = \exp\bigl(X(x)\, X(x)\bigr).} \tag{13}$$

4.2 Exact derivative law

Differentiate (13):

$$\frac{F'}{F} = \frac{d}{dx}(XX) = X'(X+1),$$

hence

$$\boxed{\frac{d}{dx}\bigl(X(x)^{X(x)}\bigr) = X(x)^{X(x)}\, X'(x)\,\bigl(X(x)+1\bigr).} \tag{14}$$

4.3 Series reconstruction when $a_0 \neq 0$

If $a_0 \neq 0$, write $X(x) = a_0(1 + U(x))$, where $U(x) = \sum_{n\geq 1} \frac{a_n}{a_0} x^n$, and

$$X(x) = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} U(x)^k.$$

Let $H(x) = X(x)X(x) = \sum_{n\geq 0} c_n x^n$. Then

$$X(x)^{X(x)} = \exp(H(x)) = \exp(c_0) \sum_{n\geq 0} b_n x^n,$$

with exact coefficient law

$$\boxed{b_n = \frac{1}{n!} B_n\bigl(1!c_1, 2!c_2, \ldots, n!c_n\bigr)} \tag{15}$$

(complete Bell polynomials). This is the Hilbert-side coefficient projection of the Tayebian object introduced next.

5 Mode Coordinates, Residue Combs, and Modular Phase Extraction

5.1 Value-mode coordinates for towers and X^X

For a positive-valued generator f, define

$$\alpha_k[f](x) := \frac{\lambda}{i\pi} \log\left(\frac{f(x)}{e^k}\right).$$

Specialize to prime-height towers $f = t_p$:

$$\alpha_k^{(p)}(x) = \frac{\lambda}{i\pi} \log\left(\frac{t_p(x)}{e^k}\right),$$

and to $f = X^X$:

$$\alpha_k^{(X)}(x) = \frac{\lambda}{i\pi} \log\left(\frac{X(x)^{X(x)}}{e^k}\right).$$

5.2 Cotangent residue-comb selector

For suitable analytic $H(z)$,

$$\sum_{k\in\mathbb{Z}} H(k) = \frac{1}{2\pi i} \oint_C H(z)\, \pi \cot(\pi z)\, dz.$$

Applying to spectral-phase terms yields selector identities for discrete mode sums; this provides a representation-independent residue comb that becomes a selector matrix after basis choice.

5.3 Modular extraction functional

For $m \geq 2$ define

$$\mathsf{Mod}_m[f](x) := \frac{m}{2\pi} \arg\left(\int_0^{m-1} \exp\Big(\frac{2\pi i}{m}(f(x)+k)\Big)\, dk\right).$$

One may apply it to values ($f = t_p$ or $f = X^X$) or to height ($f = \mathrm{slog}_x$) to move congruence observables into the linearized coordinate.

6 Tayebian Duality: A Higher Object Than Selectors

We now formalize the statement that Tayebian is higher than selector matrices by exhibiting selectors as functorial images of Tayebian objects.

6.1 Tayebian objects

Definition 1 (Tayebian object)**.** *A* Tayebian object *is a triple*

$$\mathcal{T} = (\mathcal{H}, \mathcal{L}, \Pi)$$

where:

- *$\mathcal{H}$ is a (complex) Hilbert-like representation space (states, modes, coefficient expansions).*
- *$\mathcal{L}$ is a Lagrangian/generator structure (actions, recursion/flow generators) acting on or paired with $\mathcal{H}$.*
- *Π is a pairing/projection rule that maps generator data into state amplitudes and/or induces evolution/weights on $\mathcal{H}$.*

Remark 1. *The constructions above fit this template:*

- *For towers: $\mathcal{L}$ is the iteration generator $y \mapsto x^y$ and its Abel-linearized translation in height; $\mathcal{H}$ is the value/mode description of $T(x, h)$.*
- *For X^X: $\mathcal{L}[X] = XX$ and $\mathcal{H}$ contains the coefficient/mode expansion of $\exp(XX)$.*

6.2 A Tayebian category

Definition 2 (Category **Tay**). *Define a category* **Tay** *whose objects are Tayebian triples $\mathcal{T} = (\mathcal{H}, \mathcal{L}, \Pi)$. A morphism*

$$\varphi : \mathcal{T} \to \mathcal{T}'$$

is a pair of structure-preserving maps

$$\varphi = (\varphi_H, \varphi_L)$$

such that:

1. *$\varphi_H : \mathcal{H} \to \mathcal{H}'$ is linear (Hilbert-side map),*
2. *$\varphi_L : \mathcal{L} \to \mathcal{L}'$ preserves generator composition/recursion,*
3. *Compatibility (intertwining) holds:*

$$\Pi'(\varphi_L(\ell)) \circ \varphi_H = \varphi_H \circ \Pi(\ell) \quad \textit{for all generators } \ell \in \mathcal{L}.$$

This compatibility is the abstract expression of "same object, two descriptions".

6.3 Selector algebras and matrix category

Let **Alg** be a category whose objects are operator algebras $\mathfrak{A} \subseteq \mathrm{End}(\mathcal{H})$ and whose morphisms are algebra homomorphisms. Let **Mat** be the category of matrices over a chosen field/ring (finite-dimensional truncations), or more generally bounded operators under a fixed basis choice.

7 The Selector Functor

We now define the promised functor from Tayebian objects to selector-matrix algebras.

7.1 Hilbert projection and basis choice as functorial stages

Two steps are required before "matrices" exist:

- **Hilbert projection:** forget generator data, keep the state representation space.
- **Choice of basis:** identify endomorphisms with matrices.

Definition 3 (Hilbert projection functor)**.** *Define*

$$\mathsf{H} : \mathbf{Tay} \to \mathbf{Hilb}$$

by $\mathsf{H}(\mathcal{H}, \mathcal{L}, \Pi) = \mathcal{H}$ *and* $\mathsf{H}(\varphi_H, \varphi_L) = \varphi_H$*.*

Definition 4 (Basis (gauge) choice)**.** *Fix, for each Hilbert space* $\mathcal{H}$ *in the image of* H*, a basis (or frame)* $\mathcal{B}_{\mathcal{H}}$*. This yields an identification*

$$\mathrm{End}(\mathcal{H}) \cong \mathbf{Mat}(\mathcal{B}_{\mathcal{H}}).$$

Basis choice is not intrinsic; it is a gauge. This is exactly why selectors are representation-dependent shadows.

7.2 Selector extraction from residues/modes

Selectors arise from three representation mechanisms already present:

- residue combs (e.g., $\pi \cot(\pi z)$),
- modular phase extraction,
- coordinate/mode projections (e.g., coefficient truncation, bandpass in spectral index).

These define subalgebras of $\mathrm{End}(\mathcal{H})$ generated by idempotents and projections.

Definition 5 (Selector algebra of a Tayebian object)**.** *Given a Tayebian object* $\mathcal{T} = (\mathcal{H}, \mathcal{L}, \Pi)$*, define* $\mathsf{Sel}(\mathcal{T}) \subseteq \mathrm{End}(\mathcal{H})$ *as the algebra generated by:*

1. *projectors onto chosen mode/coordinate subspaces in* $\mathcal{H}$*,*
2. *residue-comb induced selection operators (when* $\mathcal{H}$ *admits analytic continuation of indices),*
3. *modular selection operators induced by* $f \mapsto \mathsf{Mod}_m[f]$ *acting as observables on* $\mathcal{H}$*.*

7.3 Functor definition

Theorem 1 (Selector functor)**.** *There exists a functor*

$$\boxed{\mathsf{F} : \mathbf{Tay} \to \mathbf{Alg}}$$

defined by

$$\mathsf{F}(\mathcal{T}) = \mathsf{Sel}(\mathcal{T}) \subseteq \mathrm{End}(\mathsf{H}(\mathcal{T})),$$

and on morphisms $\varphi = (\varphi_H, \varphi_L)$ *by*

$$\mathsf{F}(\varphi) : \mathsf{Sel}(\mathcal{T}) \to \mathsf{Sel}(\mathcal{T}'), \qquad \mathsf{F}(\varphi)(S) = \varphi_H \, S \, \varphi_H^{-1},$$

whenever φ_H *is invertible on the relevant selected subspace (or by restriction/quotient in the non-invertible case).*

Proof. Functoriality follows from:

$$\mathsf{F}(\mathrm{id}) = \mathrm{id}, \qquad \mathsf{F}(\psi \circ \varphi) = \mathsf{F}(\psi) \circ \mathsf{F}(\varphi),$$

since conjugation respects composition. The selector generators (projections/residue/modular operators) are preserved under the intertwining compatibility built into **Tay** morphisms. □

7.4 Selector matrices as a further projection

Once a basis $\mathcal{B}_{\mathcal{H}}$ is chosen, each $S \in \mathsf{Sel}(\mathcal{T})$ corresponds to a matrix $[S]_{\mathcal{B}_{\mathcal{H}}}$. Thus we obtain a composite

$$\boxed{\mathbf{Tay} \xrightarrow{\mathsf{F}} \mathbf{Alg} \xrightarrow{\text{basis}} \mathbf{Mat}}$$

showing selector matrices are a two-step shadow of Tayebian objects.

8 Commutative Diagrams Linking Generator, State, and Selectors

8.1 Generator-to-state-to-selection

Let $\mathcal{T} = (\mathcal{H}, \mathcal{L}, \Pi)$. For a generator $\ell \in \mathcal{L}$, $\Pi(\ell)$ acts on $\mathcal{H}$. A selector $S \in \mathsf{Sel}(\mathcal{T})$ yields a selected observable/state component. The fundamental diagram is:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\Pi} & \mathrm{End}(\mathcal{H}) \\ \downarrow \varphi_L & & \downarrow (S \mapsto \varphi_H S \varphi_H^{-1}) \\ \mathcal{L}' & \xrightarrow{\Pi'} & \mathrm{End}(\mathcal{H}') \end{array}$$

with selection occurring inside $\mathrm{End}(\mathcal{H})$ after Hilbert projection.

8.2 Examples: tower height selection and residue comb

- **Tower/height:** generator is $y \mapsto x^y$, Abel-linearized to translation in h; a selector may project onto integer h-samples (or prime h-samples after additional arithmetic filtering).
- **Residue comb:** the operator corresponding to $\pi \cot(\pi z)$ selects integer-indexed modes; after basis choice it becomes a sparse selector matrix.

9 Conclusion

We extended the earlier analytic development of prime-height tetration and $X(x)^{X(x)}$ by introducing a formally higher layer: Tayebian duality, simultaneously encoding Hilbert (state/mode) and Lagrangian (generator/action) descriptions. We constructed a selector functor

$$\mathsf{F} : \mathbf{Tay} \to \mathbf{Alg}$$

mapping Tayebian objects to selector-operator algebras on Hilbert projections, and we showed selector matrices arise only after an additional basis (gauge) choice. This functorial formulation fixes the hierarchy: selectors are not competitors to Tayebian; they are representation-dependent images of Tayebian objects.

A Appendix A: Summary of Key Exact Identities

Towers

$$t_1(x) = x, \quad t_{n+1}(x) = x^{t_n(x)}, \quad \ell_{n+1}(x) = t_n(x) \log x.$$

$$t'_{n+1}(x) = t_{n+1}(x) \left(t'_n(x) \log x + \frac{t_n(x)}{x} \right), \quad D_{n+1}(x) = D_n(x)\ell_{n+1}(x) + \frac{t_n(x)}{x}.$$

$$dx = \frac{du}{uD_p(\tau_p(u))} \quad (u = t_p(x)).$$

Continuous height

$$A_x(x^y) = A_x(y) + 1, \quad T(x,h) = A_x^{-1}(A_x(1) + h), \quad \operatorname{slog}_x(y) = A_x(y) - A_x(1).$$

Infinite polynomial substitution

$$X^X = \exp(XX), \qquad (X^X)' = X^X\, X'(X+1).$$

B Appendix B: Finite-Dimensional Truncation and Matrix Realization

For practical finite-dimensional selector matrices, choose a truncation of $\mathcal{H}$ (e.g. coefficient cutoff $n \leq N$ or mode cutoff $k \leq K$). Then

$$\mathsf{Sel}(\mathcal{T}) \subseteq \operatorname{End}(\mathcal{H}_N) \cong \mathbf{Mat}_{N\times N}.$$

Different truncations yield a directed system of algebras whose inductive limit recovers the infinite-dimensional operator algebra.